Final Solutions

1. Suppose that the equation $F(x,y,z)=0$ implicitly defines each of the three variables x, y, and z as functions of the other two:

$$z=f(x,y), y=g(x,z), x=h(y,z).$$

If F is differentiable and F_x, F_y, and F_z are all nonzero, show that [10]

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=-1.$$

Solution. From a Theorem on implicit differentiation done in class, we have that

$$\frac{\partial z}{\partial x}=f_x=-\frac{F_x}{F_z}$$
$$\frac{\partial x}{\partial y}=h_y=-\frac{F_y}{F_x}$$
$$\frac{\partial y}{\partial z}=g_z=-\frac{F_z}{F_y}$$

From these three equations, we have

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=-1.$$

2. Can we have a differentiable scalar field $f:\mathbb{R}^2\to\mathbb{R}$ satisfying both of the following conditions?

(a) The partial derivatives $f_x(0,0)=f_y(0,0)=0$, and
(b) the directional derivative $f'(0;i+j)=3$.

Solution. Since $f_x(0,0)=f_y(0,0)=0$,

$$\nabla f(0,0)=f_x(0,0)\,i+f_y(0,0))\,j=0.$$

But the fact that f is differentiable would imply that

$$f'(0;i+j)=\nabla f(0,0)\cdot(i+j)=0,$$

which contradicts condition (b).

3. A cylinder whose equation is $y = f(x)$ is tangent to the surface $z^2 + 2xz + y = 0$ at all points common to the two surfaces. Find $f(x)$.

   Solution. Let $F(x, y, z) = f(x) - y$, $G(x, y, z) = z^2 + 2xz + y$, let S denote the set of point common to the two surfaces. Since these two surfaces are tangent to each other at each $(x, y, z) \in S$, we have that

   $$\nabla f(x, y, z) \cdot \nabla g(x, y, z) = 0, \text{ for } (x, y, z) \in S.$$

   In other words,

   $$(f'(x), -1, 0) \cdot (2z, 1, 2z + 2x) = 0, \text{ for } (x, y, z) \in S,$$

   where $f'(x) = f_x$. Upon simplification, we have the ordinary differential equation

   $$f'(x) = \frac{1}{2z}, \text{ for } (x, y, z) \in S.$$

   In S, we must have

   $$f(x) = -z^2 + 2xz.$$

   Solving for z in terms of x from this quadratic equaation, we obtain

   $$z = -x \pm \sqrt{x^2 - f(x)}.$$

   Therefore, f is solution to the ordinary differential equation

   $$f'(x) = \frac{dy}{dx} = \frac{1}{-x \pm \sqrt{x^2 - y}}.$$

4. Find three positive numbers whose sum is hundred and whose product is a maximum.

   Solution. Let x, y, z denote the three numbers. We need to maximize $f(x, y, z) = xyz$, subject to the constraint $g(x, y, z) = x + y + z = 100$. Using the method of Lagrange's multipliers, we obtain the following system of equations (for some nonzero $\lambda \in \mathbb{R}$)

   $$\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \\ x + y + z &= 100. \end{aligned}$$

From this system of equations, we obtain the equivalent system

$$\lambda x = \lambda y = \lambda z$$
$$x + y + z = 100.$$

Since $\lambda \neq 0$, we have that $x = y = z$, which upon substitution in $x + y + z = 100$ yields $x = y = z = \frac{100}{3}$. It is easy to see that this is a maximum, as $f(98, 1, 1) < f(\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$.

5. Use Stokes' Theorem to evaluate $\int_c (y+\sin x)\, dx + (z^2+\cos y)\, dy + x^3\, dz$, where C is the curve $r(t) = (\sin t, \cos t, \sin 2t)$, $t \in [0, 2\pi]$.

   Solution. Let $F(x, y, z) = Mi + Nj + Pk = (y + \sin x)\, i + (z^2 + \cos y)\, j + x^3\, j$. Clearly, the components of F: $M = y + \sin x, N = z^2 + \cos y, P = x^3$ have continuous first partials everywhere in $\mathbb{R}^3$. Furthermore, $r(t)$ is a simple closed $(r(0) = r(2\pi))$ and smooth curve that lies on the smooth surface $z = 2xy$. Let S denote the part of the surface $z = 2xy$ bounded by $r(t)$. Note that the projection of S onto the xy-plane is the unit disk D centered at origin. Also, C is traversed clockwise (when viewed from above) and S is oriented downward. For this S and C, the hypotheses of Stokes' Theorem are satisfied.

   By the Stokes' Theorem, we have that

$$\int_C F \cdot dr = -\iint_S (\nabla \times F) \cdot n\, d\sigma.$$

   By a simple calculation, $n = \frac{2y}{\sqrt{4x^2+4y^2+1}}\, i + \frac{2x}{\sqrt{4x^2+4y^2+1}}\, j - \frac{1}{\sqrt{4x^2+4y^2+1}}\, k$ and $F = -2z\, i - 3x^2\, j - k$. Therefore,

$$\begin{aligned} -\iint_S (\nabla \times F) \cdot n\, d\sigma &= -\iint_D \frac{8xy^2 + 6x^3 - 1}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1} dA \\ &= -\iint_D (8xy^2 + 6x^3 - 1)\, dA \\ &= -\int_0^{2\pi} \int_0^1 (8r^3 \cos\theta \sin^2\theta + 6r^3 \cos^3\theta - 1) r\, dr d\theta \\ &= \pi. \end{aligned}$$

6. Use the Gauss' Divergence Theorem to evaluate $\iint_S F \cdot n\, d\sigma$, where

$$F(x, y, z) = z^2 x\, i + (\frac{y^3}{3} + \tan z)\, j + (x^2 z + y^2)\, k$$

and S is top half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution. Let S_1 denote the unit disk in the xy-plane centered at origin. Then $S' = S \cup S_1$ is a piecewise smooth closed surface enclosing a region $E \in \mathbb{R}^3$. Also, the components of $F : M = z^2 x, N = \frac{y^3}{3} + \tan z, P = x^2 z + y^2$ have continuous first partials in an open set containing $D \cup S'$ (as $z \neq \pi/2$ anywhere in $D \cup S'$). Therefore, the hypotheses of the Gauss' Divergence Theorem are satisfied.

By the Gauss' Divergence Theorem, we have that

$$\iint_{S'} F.n\, d\sigma = \iint_{S} F.n\, d\sigma + \iint_{S_1} F.n\, d\sigma = \iiint_E \nabla \cdot F\, dV.$$

For S_1, we have $n = -k$ and $z = 0$, and so

$$\begin{aligned} \iint_{S_1} F.n\, d\sigma &= \iint_{S_1} (-y^2)\, dA \\ &= \int_0^{2\pi} \int_0^1 r^2 (\sin^2 \theta) r\, dr d\theta \\ &= -\frac{\pi}{4} \end{aligned}$$

By a simple calculation $\nabla \cdot F = x^2 + y^2 + z^2$, and we have

$$\begin{aligned} \iiint_E \nabla \cdot F\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \frac{2\pi}{5} \end{aligned}$$

Finally,

$$\iint_S F.n\, d\sigma = \iiint_E \nabla \cdot F\, dV - \iint_{S'} F.n\, d\sigma = \frac{13\pi}{20}.$$

7. If $F(x, y) = -\left(\frac{y}{x^2+y^2}\right) i + \left(\frac{x}{x^2+y^2}\right) j$, show that $\int_C F \cdot dr = 2\pi$ along any counterclockwise oriented simple closed curve that encloses the origin.

Solution. Let C be an arbitrary simple closed curve that encloses the origin. Let C' be a counterclockwise oriented circle with center the origin and radius a, where a is chosen small enough so that C' lies inside C. Let D be the region bounded by C and C'. The the positively oriented boundary of D is $C \cup (-C')$.

By the Circulation-Curl form of the Green's Theorem, we have that

$$\begin{aligned}\int_C M\,dx + N\,dy + \int_{-C'} M\,dx + N\,dy &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right) dA \\ &= \iint_D \left[\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}\right] dA \\ &= 0.\end{aligned}$$

In other words,

$$\int_C M\,dx + N\,dy = \int_{C'} M\,dx + N\,dy.$$

Using the polar coordinates $x = a\cos\theta$ and $y = a\sin\theta$, we have that

$$\begin{aligned}\int_{C'} M\,dx + N\,dy &= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt \\ &= 2\pi.\end{aligned}$$

8. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field and $f, g : \mathbb{R}^3 \to \mathbb{R}$ are scalar fields, show that

   (a) $\nabla \cdot (\nabla \times F) = 0$.

   (b) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

Solution. (a) Let $F = M\,i + N\,j + P\,k$. Then

$$\nabla \times F = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) i + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) j + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) k,$$

and hence

$$\begin{aligned}\nabla \cdot (\nabla \times F) &= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} \\ &= 0.\end{aligned}$$

(b) By the definition of the gradient, we have that

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \left(\frac{f}{g}\right)_x i + \left(\frac{f}{g}\right)_y j + \left(\frac{f}{g}\right)_z k \\
&= \left(\frac{f_x g - f g_x}{g^2}\right) i + \left(\frac{f_y g - f g_y}{g^2}\right) j + \left(\frac{f_z g - f g_z}{g^2}\right) k \\
&= \frac{g(f_x i + f_y j + f_z k) - f(g_x i + g_y j + g_z k)}{g^2} \\
&= \frac{g\nabla f - f\nabla g}{g^2}.
\end{aligned}$$

9. Solve the differential equation

$$(y^3 + xy^2 + y)\,dx + (x^3 + x^2 y + x)\,dy = 0.$$

Solution. From the equation, we have

$$P_y = 3y^2 + 2xy + 1,\ Q_x = 3x^2 + 2xy + 1.$$

Clearly, the equation is not exact. We use an integrating factor of the form $h = h(u)$, where $u = xy$.

Let $F(u) = \dfrac{P_y - Q_x}{yQ - xP} = \dfrac{3(y^2 - x^2)}{xy(x^2 - y^2)} = -\dfrac{3}{u}$. Then the integrating factor

$$h(u) = e^{-\int \frac{3}{u} du} = u^{-3} = \frac{1}{x^3 y^3}.$$

Multiplying the differential equation by the integrating factor, we obtain the following exact differential equation

$$\left(\frac{1}{x^3} + \frac{1}{x^2 y} + \frac{1}{x^3 y^2}\right) dx + \left(\frac{1}{y^3} + \frac{1}{xy^2} + \frac{1}{x^2 y^3}\right) dy = 0.$$

We denote these new coefficients of dx and dy by P' and Q' respectively. We choose $x_0 = y_0 = 1$ so that the rectangle with vertices x, x_0, y, y_0 lies entirely in the region

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0 \text{ and } y \neq 0\},$$

where P', Q', and all of their first partial exist and are continuous.

A solution to this exact differential equation is given by

$$f(x, y) = \int_1^x \left(\frac{1}{x^3} + \frac{1}{x^2 y} + \frac{1}{x^3 y^2}\right) dx + \int_1^y \left(\frac{1}{y^3} + \frac{1}{y^2} + \frac{1}{y^3}\right) dy = 0,$$

that is

$$\frac{1}{2x^2} + \frac{1}{xy} + \frac{1}{2x^2y^2} + \frac{1}{2y^2} + \frac{1}{y} + \frac{1}{2y^2} = c,$$

which upon simplification yields the solution

$$y^2 + 2xy + 2x^2 + 2x^2y + 1 = kx^2y^2.$$

10. **(Bonus)** Show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Solution. Let $I = \int_0^\infty e^{-x^2}\,dx$. Then

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

(This is due to the Fubini's Theorem.)

Converting this double integral into polar coordinates, we have

$$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta.$$

We now use the substitution $r^2 = u$, to obtain

$$\begin{aligned} I^2 &= \frac{1}{2}\int_0^{\pi/2}\int_0^\infty e^{-u}\,du\,d\theta \\ &= \frac{\pi}{4}. \end{aligned}$$

Therefore, $I = \frac{\sqrt{\pi}}{2}$.